

A SPECIALITY THEOREM FOR CURVES IN \mathbb{P}^5 CONTAINED IN NOETHER-LEFSCHETZ GENERAL FOURFOLDS

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ABSTRACT. Let $C \subset \mathbb{P}^r$ be an integral projective curve. We define the speciality index $e(C)$ of C as the maximal integer t such that $h^0(C, \omega_C(-t)) > 0$, where ω_C denotes the dualizing sheaf of C . In the present paper we consider $C \subset \mathbb{P}^5$ an integral degree d curve and we denote by s the minimal degree for which there exists a hypersurface of degree s containing C . We assume that C is contained in two smooth hypersurfaces F and G , with $\deg(F) = n > k = \deg(G)$. We assume additionally that F is Noether-Lefschetz general, i.e. that the 2-th Néron-Severi group of F is generated by the linear section class. Our main result is that in this case the speciality index is bounded as $e(C) \leq \frac{d}{snk} + s + n + k - 6$. Moreover equality holds if and only if C is a complete intersection of $T := F \cap G$ with hypersurfaces of degrees s and $\frac{d}{snk}$.

Keywords: Complex projective curve; speciality index; arithmetic genus; linkage; Castelnuovo - Halphen Theory.

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1. INTRODUCTION

Let $C \subset \mathbb{P}^r$ be an integral projective curve. We define the *speciality index* $e(C)$ of C as the maximal integer t such that $h^0(C, \omega_C(-t)) > 0$, where ω_C denotes the dualizing sheaf of C . The speciality index of a space curve is a fundamental invariant which turned out to be crucial in many issues of projective geometry. For instance, in the papers [7], [8] and [9], such an invariant has been proved to be very useful in the study of projective manifolds of codimension 2.

In [13] Gruson and Peskine prove the following theorem concerning the speciality index of an integral space curve (see also [14]):

Theorem 1.1 (Speciality Theorem). *Let $C \subset \mathbb{P}^3$ be an integral degree d curve not contained in any surface of degree $< s$. Then we have:*

$$e(C) \leq \frac{d}{s} + s - 4.$$

Moreover equality holds if and only if C is a complete intersection of surfaces of degrees s and $\frac{d}{s}$.

In our previous work [2], we prove an extension of this theorem to curves in \mathbb{P}^5 :

Theorem 1.2. [2, Theorem B] *Let $C \subset \mathbb{P}^5$ be an integral degree d curve not contained in any surface of degree $< s$, in any threefold of degree $< t$, and in any*

fourfold of degree $< u$. Assume $d \gg s \gg t \gg u \geq 1$. Then we have:

$$e(C) \leq \frac{d}{s} + \frac{s}{t} + \frac{t}{u} + u - 6.$$

Moreover equality holds if and only if C is a complete intersection of hypersurfaces of degrees u , $\frac{t}{u}$, $\frac{s}{t}$ and $\frac{d}{s}$.

Unfortunately, it seems hard to find a generalization of Gruson-Peskine Speciality Theorem without the assumptions $d \gg s \gg t \gg u \geq 1$ and to prove a sharp version of the Speciality Theorem for curves in \mathbb{P}^5 .

In this paper we adopt a somewhat different strategy and prove a sharp version of the Speciality Theorem for curves in \mathbb{P}^5 under the assumption that the curve is contained in a smooth hypersurface with a nice behaviour from the point of view of Noether-Lefschetz theory (compare with Definition 2.2). More precisely, what we are going to do is to assume that C is contained in a smooth hypersurface having the 2-th Néron-Severi group generated by the linear section class. The main results of this paper are collected in the following Theorem.

Theorem 1.3. *Let $C \subset \mathbb{P}^5$ be an integral degree d curve. Assume that C is contained in two smooth hypersurfaces F and G , with $\deg(F) = n > k = \deg(G)$. Assume additionally that F is Noether-Lefschetz general, i.e. that the 2-th Néron-Severi group of F is generated by the linear section class.*

- (1) *If C is not contained in any hypersurface of degree $< s$, then we have:*

$$e(C) \leq \frac{d}{snk} + s + n + k - 6.$$

- (2) *If C is contained in a hypersurface of degree $s < k$, then the inequality above still holds true. Moreover, equality holds if and only if C is a complete intersection of $T := F \cap G$ with hypersurfaces of degrees s and $\frac{d}{snk}$.*

Theorem 1.3 turned out to be a consequence of much more general results stated in Theorem 3.2 and Theorem 4.1. They show that a sort of Speciality Theorem holds true for Cohen-Macaulay subschemes $X \subset T$ of codimension 2 in any arithmetically Cohen-Macaulay and factorial variety T .

2. NOTATIONS AND PRELIMINARY RESULTS

In order to prove Theorem 1.3, in this section we gather some known properties and results, mainly borrowed from our previous works [3], [4] and [6].

Notations 2.1. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of dimension $2i \geq 2$. Denote by $NS_i(X; \mathbb{Z})$ be the i -th Néron-Severi group of X , i.e. the image of the cycle map:

$$NS_i(X; \mathbb{Z}) := \text{Im}(A_i(X) \rightarrow H_{2i}(X; \mathbb{Z}) \cong H^{2i}(X; \mathbb{Z})).$$

Definition 2.2. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of dimension $2i \geq 2$. By the Lefschetz hyperplane section theorem we know that the homology group $H_{2i}(X; \mathbb{Z}) \cong H^{2i}(X; \mathbb{Z})$ is free. We will say that X is *Noether-Lefschetz general* if the rank of $NS_i(X; \mathbb{Z})$ is one. In this case, the Lefschetz hyperplane section theorem also implies that $NS_i(X; \mathbb{Z})$ is generated by the linear section class H^i .

In [3], it can be found a proof for the following result:

Theorem 2.3. [3, Theorem 1.1] *Let F and G be smooth hypersurfaces in \mathbb{P}^{2m+1} , with $\deg(F) = n > k = \deg(G)$, and set $X = F \cap G$. If F is Noether-Lefschetz general then $\text{rk} NS_m(X) = 1$, and $NS_m(X)$ is generated by the linear section class.*

Notations 2.4. (1) Let $Q \subseteq \mathbb{P}^n$ be an irreducible, reduced, non-degenerate projective variety of dimension $m+1$, with isolated singularities. Let Q_t be a general hyperplane section of Q . Let $U \subset \check{\mathbb{P}}^n$ be the open subset parametrizing smooth hyperplane sections of Q . The fundamental group $\pi_1(U)$ acts via monodromy on both $H^m(Q_t; \mathbb{Z})$ and $H^m(Q_t; \mathbb{Q})$. We denote by

$$H^m(Q_t; \mathbb{Q}) = \mathbb{I} \oplus \mathbb{V}$$

the orthogonal decomposition given by the monodromy action on the cohomology of Q_t , where \mathbb{I} denotes the invariant subspace.

(2) Denote by

$$i_k^* : H_{k+2}(Q; \mathbb{Z}) \rightarrow H^{2n-k}(Q_t; \mathbb{Z})$$

the map obtained composing the Gysin map $H_{k+2}(Q; \mathbb{Z}) \rightarrow H_k(Q_t; \mathbb{Z})$ with Poincaré duality $H_k(Q_t; \mathbb{Z}) \cong H^{2n-k}(Q_t; \mathbb{Z})$.

In [4], it can be found a proof for the following results:

Theorem 2.5. [4, Theorem 3.1] *With notations as in 2.4, the vector subspace $\mathbb{V} \subset H^m(Q_t; \mathbb{Q})$ is generated, via monodromy, by standard vanishing cycles.*

Corollary 2.6. [4, Corollary 3.7] *The vector subspace $\mathbb{V} \subset H^m(Q_t; \mathbb{Q})$ is irreducible via monodromy action.*

The results 2.5 and 2.6 concern rational cohomology. In the paper [6] they are used to prove similar results concerning integral cohomology:

Theorem 2.7. [6, Theorem 2.1] *With notations as in 2.4, the following properties hold true.*

- (1) *For any integer $m < k \leq 2m$ the map i_k^* is an isomorphism, the map i_m^* is injective with torsion-free cokernel, and $H_{m+2}(Q; \mathbb{Z}) \cong \mathbb{I}$ via i_m^* .*
- (2) *For any even integer $m < k = 2i \leq 2m$ the map $i_k^* \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism*

$$NS_{i+1}(Q; \mathbb{Q}) \cong NS_i(Q_t; \mathbb{Q}).$$

- (3) If $k = 2i = m$ and the orthogonal complement \mathbb{V} of $\mathbb{I} \otimes_{\mathbb{Z}} \mathbb{Q}$ in $H^n(Q_t; \mathbb{Q})$ is not of pure Hodge type $(m/2, m/2)$, then $NS_i(Q_t; \mathbb{Z}) \subseteq \mathbb{I}$, and the map $i_n^* \otimes_{\mathbb{Z}} \mathbb{Q}$ induces an isomorphism $NS_{i+1}(Q; \mathbb{Q}) \cong NS_i(Q_t; \mathbb{Q})$.

One of the main ingredients of the proof of Theorem 1.3 is the following Proposition.

Proposition 2.8. *Let $F \subset \mathbb{P}^5$ be a Noether-Lefschetz general hypersurface and let $G \subset \mathbb{P}^5$ be a smooth hypersurface with $k := \deg G < d := \deg F$. Define $T := F \cap G$. Then we have:*

- (1) *the threefold T is factorial with isolated singularities;*
- (2) *if $\deg T \geq 4$ then the general hyperplane section $S := H \cap T$ is a Noether-Lefschetz general surface.*

Proof. (1) The threefold T has at worst finitely many singularities by [10, Proposition 4.2.6]. Furthermore, T is factorial by Theorem 2.3.

(2) Combining Theorem 2.5, Corollary 2.6 and Theorem 2.7, the proof runs similarly as the classical one (compare with the proof of [5, Theorem 3.2]). Indeed, denote by $U \subset \mathbb{P}^5$ the affine open subset parametrizing smooth hyperplane sections of T . The fundamental group $\pi_1(U)$ acts via monodromy on both $H^2(S; \mathbb{Z})$ and $H^2(S; \mathbb{Q})$. As in 2.4, consider the orthogonal decomposition $H^2(S; \mathbb{Q}) = \mathbb{I} \perp \mathbb{V}$, where \mathbb{I} is the $\pi_1(U)$ -invariant cohomology (compare also with [5, Notations 3.1 (ii)]). By Theorem 2.5 and Corollary 2.6 we know that the vanishing cohomology \mathbb{V} is a $\pi_1(U)$ -irreducible module generated by the standard vanishing cycles. On the other hand, Theorem 2.7 implies that the $\pi_1(U)$ -invariant part of $H^2(S; \mathbb{Z}) \simeq H_2(S; \mathbb{Z})$ is the image of the Gysin map:

$$(1) \quad \mathbb{I} \cap H_2(S; \mathbb{Z}) = \text{Im}(H_4(T; \mathbb{Z}) \xrightarrow{\cap u} H_2(S; \mathbb{Z}))$$

(here $u \in H^2(T, T - S; \mathbb{Z})$ denotes the orientation class [12, §19.2]). By point (1) T is factorial, hence the subspace \mathbb{I} is generated by the hyperplane class. But then \mathbb{V} is not of pure Hodge type because $\deg T \geq 4$. By irreducibility, the image of $NS_1(S; \mathbb{Z})$ in \mathbb{V} vanishes. This implies that the Néron-Severi group $NS_1(S; \mathbb{Z})$ is $\pi_1(U)$ -invariant and (1) says that S is Noether-Lefschetz general. \square

3. PROOF OF THEOREM 1.3 (2)

Definition 3.1. Let X be a Cohen-Macaulay projective scheme. We define the *speciality index* e_X of X as the maximal integer t such that $h^0(X, \omega_X(-t)) > 0$, where ω_X denotes the dualizing sheaf of X .

The proof of Theorem 1.3 (2) rests on the following much more general result:

Theorem 3.2 (Speciality theorem for aCM varieties). *Let $T \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay (aCM for short), factorial and subcanonical variety with $\dim T = m \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$.*

Let $G \subset T$ be an integral divisor. Since T is factorial and aCM, we have $G = \tilde{G} \cap T$ with $\tilde{G} \subset \mathbb{P}^n$ a projective hypersurface of some degree g . Let $X \subset G$ be a Cohen-Macaulay scheme of codimension two in T .

Then

$$e_X \leq \frac{\deg(X)}{\deg(T)g} + g + t$$

and the equality holds iff X is a complete intersection $X = T \cap \tilde{G} \cap H$, with $\deg(H) = \frac{\deg(X)}{\deg(T)g}$.

Proof. Consider a general hypersurface P of degree $p \gg 0$ containing X . Denote by Y the scheme $T \cap \tilde{G} \cap P$ which we are going to consider as a complete intersection in T . Following Peskine-Szpiro [17], we consider the scheme R residual of X with respect to Y (compare also with [11, §2]).

The Noether Linkage Sequence [11, Proposition 2.3] inside T looks like

$$(2) \quad 0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_R \rightarrow \omega_X \otimes \omega_Y^{-1} \rightarrow 0,$$

and can be written as

$$(3) \quad 0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_R \rightarrow \omega_X(-t - g - p) \rightarrow 0$$

(all the ideal sheaves are meant to be defined in T). Recall that

$$(4) \quad h^0(\omega_X(-e)) \neq 0$$

($e := e_X$). Since T is aCM and Y is a complete intersection in T of type (g, p) , the short exact sequence

$$0 \rightarrow \mathcal{O}_T(-g - p) \rightarrow \mathcal{O}_T(-g) \oplus \mathcal{O}_T(-p) \rightarrow \mathcal{I}_Y \rightarrow 0$$

implies

$$\cdots \rightarrow H^1(\mathcal{O}_T(l - g) \oplus \mathcal{O}_T(l - p)) \rightarrow H^1(\mathcal{I}_Y(l)) \rightarrow H^2(\mathcal{O}_T(l - g - p)) \rightarrow \cdots \quad \forall l$$

hence

$$(5) \quad h^1(\mathcal{I}_Y(t + g + p - e)) = 0.$$

Combining (3), (4) and (5), we see that there exists a hypersurface S of degree $s = t + g + p - e$ containing R and not containing Y . But G is integral and $Y' = G \cap S$ is a complete intersection, in T , containing R . Set $Y' = R \cup R'$ the corresponding, possibly algebraic, linkage. But then

$$\deg(R') + \deg(R) = \deg(T)gs, \quad \deg(X) + \deg(R) = \deg(T)gp$$

and by a simple computation, we find

$$\deg(R') = \deg(X) - \deg(T)g(e - t - g) \geq 0$$

and the first statement follows.

Suppose now the equality holds. Then we have

$$\deg(X) = \deg(T)g(e - t - g) = \deg(T)g(p - s).$$

and the scheme R' is empty. Furthermore, we have that $R = Y' = G \cap S$ is a complete intersection with $\omega_R \simeq \mathcal{O}_R(t + g + s)$.

Coming back to the Noether Linkage Sequence (2)

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_X \rightarrow \omega_R \otimes \omega_Y^{-1} \rightarrow 0$$

we find

$$(6) \quad 0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_R(s-p) \rightarrow 0.$$

Similarly as above, the short exact sequence

$$0 \rightarrow \mathcal{O}_T(-g-s) \rightarrow \mathcal{O}_T(-g) \oplus \mathcal{O}_T(-s) \rightarrow \mathcal{I}_R \rightarrow 0$$

implies

$$\cdots \rightarrow H^1(\mathcal{O}_T(l-g) \oplus \mathcal{O}_T(l-s)) \rightarrow H^1(\mathcal{I}_Y(l)) \rightarrow H^2(\mathcal{O}_T(l-g-s)) \rightarrow \cdots \quad \forall l$$

and

$$h^1(\mathcal{I}_R(p-s)) = 0.$$

Hence there is a hypersurface H of degree $h = p-s$ containing X and not containing Y . Finally, since G is integral and $\deg(X) = \deg(T)g(p-s)$ we conclude that $X = G \cap H$. \square

Proof of Theorem 1.3 (2). It suffices to apply Theorem 3.2 to the complete intersection $T := F \cap G$, which is aCM with $\dim T = 3$ and $\omega_T \simeq \mathcal{O}_T(n+k-6)$, and factorial in view of Proposition 2.8. \square

4. PROOF OF THEOREM 1.3 (1)

The proof of Theorem 1.3 (1) rests on the following much more general result:

Theorem 4.1. *Let $T \subset \mathbb{P}^n$ be an aCM, factorial variety with $\dim T = m \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$. Assume moreover that T is smooth in codimension 2 and that the very general surface section of T is factorial. Let $X \subset T$ be a C.M. subscheme of codimension 2 which is generically complete intersection. If $h^0(\mathcal{I}_{X,T}(h-1)) = 0$ and $h > 0$ then*

$$e_X \leq \frac{\deg(X)}{\deg(T)h} + h + t.$$

The main idea in the proof of 4.1, which goes back to the work of Hartshorne, is to construct a rank two reflexive sheaf on T having a section vanishing in X (see e.g. [16] and [1]).

In order to prove Theorem 4.1, we need some preliminary results. We recall the following result of R. Hartshorne:

Lemma 4.2. [16, Proposition 1.3] *Let T be a normal scheme and let \mathcal{F} be a coherent sheaf defined on T . Then \mathcal{F} is reflexive iff*

- (1) \mathcal{F} is torsion-free;
- (2) $\forall x \in T, \dim \mathcal{O}_x \geq 2 \implies \text{depth} \mathcal{F}_x \geq 2$.

For the sake of completeness, we give a short proof of the following (maybe well known) result.

Lemma 4.3. *Let $T \subset \mathbb{P}^n$ be an aCM scheme such that $m := \dim T \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$. Let $X \subset T$ be a Cohen-Macaulay subscheme of codimension 2. Then we have:*

$$\text{Ext}_T^1(\mathcal{I}_{X,T}(c), \mathcal{O}_T) \simeq H^0(X, \omega_X(-c-t)), \quad \forall c \in \mathbb{Z}.$$

Proof. By applying the functor $\text{Hom}_T(\cdot, \mathcal{O}_T)$ to the short exact sequence

$$0 \rightarrow \mathcal{I}_{X,T}(c) \rightarrow \mathcal{O}_T(c) \rightarrow \mathcal{O}_X(c) \rightarrow 0$$

we find

$$\begin{aligned} \text{Ext}_T^1(\mathcal{O}_T(c), \mathcal{O}_T) &\rightarrow \text{Ext}_T^1(\mathcal{I}_{X,T}(c), \mathcal{O}_T) \rightarrow \\ &\rightarrow \text{Ext}_T^2(\mathcal{O}_X(c), \mathcal{O}_T) \rightarrow \text{Ext}_T^2(\mathcal{O}_T(c), \mathcal{O}_T). \end{aligned}$$

By Serre Duality, $\omega_T \simeq \mathcal{O}_T(t)$ implies

$$\text{Ext}_T^i(\mathcal{O}_T(c), \mathcal{O}_T) \simeq H^{m-i}(\mathcal{O}_T(-c-t)) = 0, \quad i = 1, 2$$

where the last equality follows from the hypothesis that T is aCM of dimension ≥ 3 . Again by Serre Duality we have:

$$\begin{aligned} \text{Ext}_T^1(\mathcal{I}_{X,T}(c), \mathcal{O}_T) &\simeq \text{Ext}_T^2(\mathcal{O}_X(c), \omega_T(-t)) \simeq \\ &\simeq H^{m-2}(T, \mathcal{O}_X(c+t)) \simeq H^{m-2}(X, \mathcal{O}_X(c+t)) \simeq H^0(X, \omega_X(-c-t)). \end{aligned}$$

□

Proposition 4.4. *Let $T \subset \mathbb{P}^n$ be an aCM variety such that $m := \dim T \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$. We assume additionally that T is smooth in codimension 2. For any pair (X, ξ) with:*

- $X \subset T$ a Cohen-Macaulay, generically complete intersection subscheme of codimension two in T ,
- $\xi \in H^0(\omega_X(-t-c))$ generating almost everywhere,

there exists a rank two reflexive sheaf \mathcal{F} on T , with $c_1(\mathcal{F}) = cH$, $c_2(\mathcal{F}) = [X]$ (the fundamental cycle of X) and such that

$$(7) \quad 0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{F} \rightarrow \mathcal{I}_{X|T}(c) \rightarrow 0.$$

Proof. The assertion concerning the Chern classes follows trivially from the rest of the statement so it suffices to prove the existence of a sequence like (7), with \mathcal{F} reflexive.

The existence of a sequence like (7) follows directly from Lemma 4.3. Since T is Cohen-Macaulay and smooth in codimension 2, it is also normal by Serre's criterion. Then we may apply Lemma 4.2 in order to prove the reflexivity of \mathcal{F} . Further, since T is Cohen-Macaulay, both \mathcal{O}_T and $\mathcal{I}_{X,T}(c)$ are torsion-free hence we only need to prove the second condition of Lemma 4.2. Fix a point x of codimension ≥ 3 and denote by \mathbb{K} the residue field at x . Applying the functor $\text{Hom}_{\mathcal{O}_x}(\mathbb{K}, \cdot)$ to the sequence

$$0 \rightarrow \mathcal{I}_{x,X|T} \rightarrow \mathcal{O}_{x,T} \rightarrow \mathcal{O}_{x,X} \rightarrow 0$$

and recalling that both T and X are Cohen-Macaulay we have:

$$(8) \quad \text{Ext}_{\mathcal{O}_x}^i(\mathbb{K}, \mathcal{I}_{x,X|T}) = 0, \quad i \leq 2.$$

Applying the functor $\text{Hom}_{\mathcal{O}_x}(\mathbb{K}, \cdot)$ to the sequence

$$0 \rightarrow \mathcal{O}_{x,T} \rightarrow \mathcal{F}_x \rightarrow \mathcal{I}_{x,X|T}(c) \rightarrow 0$$

we see that the vanishing (8) implies:

$$\forall x \in X, \quad \dim \mathcal{O}_x \geq 3 \implies \text{depth} \mathcal{F}_x \geq 2.$$

In order to conclude we need to prove:

$$\forall x \in X, \quad \dim \mathcal{O}_x = 2 \implies \text{depth} \mathcal{F}_x \geq 2.$$

What we are going to do is to prove that \mathcal{F}_x is a free module of rank two over \mathcal{O}_x , for any $x \in X$ such that $\dim \mathcal{O}_x = 2$. In order to do this, we prove that \mathcal{F}_x has homological dimension 0 ([19, IV]). Since T is smooth in codimension 2, $\forall x \in X$ of codimension 2 the local ring \mathcal{O}_x is regular of dimension 2. So it suffices to prove that

$$(9) \quad \mathcal{E}xt_T^1(\mathcal{F}, \mathcal{O}_T)_x = \mathcal{E}xt_T^2(\mathcal{F}, \mathcal{O}_T)_x = 0.$$

From the sequence (7) we see that $dh(\mathcal{F}_x) \leq dh(\mathcal{I}_{X|T}) = 1$, the first inequality coming from ([19, IV p. 28]) and the last equality coming from the fact that $\mathcal{I}_{X|T}$ is complete intersection at x . So, in order to prove (9) we are left to show that $\mathcal{E}xt_T^1(\mathcal{F}, \mathcal{O}_T)_x = 0$. Applying $\text{Hom}_T(\cdot, \mathcal{O}_T(c))$ to the sequence (7) we get:

$$(10) \quad 0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{F}^*(c) \rightarrow \mathcal{O}_T \xrightarrow{\xi} \omega_X(-t) \rightarrow \mathcal{E}xt_T^1(\mathcal{F}, \mathcal{O}_T(c)) \rightarrow 0$$

where we have taken into account the isomorphism $\mathcal{E}xt_T^1(\mathcal{I}_{X,T}, \mathcal{O}_T) \simeq \omega_X(-t)$, which again follows from the fact that both T and X are Cohen-Macaulay and $\omega_T \simeq \mathcal{O}_T(t)$. Since T is smooth in codimension 2, $\forall x \in X$ of codimension 2 the local ring \mathcal{O}_x is regular of dimension 2. Furthermore, since ξ generates almost everywhere and X is generically complete intersection, the fourth map of the sequence (10) is an isomorphism at x hence $\mathcal{E}xt_T^1(\mathcal{F}, \mathcal{O}_T(c))_x \simeq 0$ and \mathcal{F}_x is a free module of rank two over \mathcal{O}_x . \square

Lemma 4.5. *Let $C \subset \mathbb{P}^n$ be a smooth variety and E a rank two vector bundle on C having a section vanishing in the right dimension. If $c_1(E) < 0$ then $h^0(E) = 1$ and $h^0(E(-m)) = 0$, $\forall m > 0$.*

Proposition 4.6. *Let $T \subset \mathbb{P}^n$ be an aCM, factorial variety such that $m := \dim T \geq 3$ and $\omega_T \simeq \mathcal{O}_T(t)$. We assume additionally that T is smooth in codimension 2 and that the general hyperplane section of T is factorial. Let \mathcal{F} be a normalized (i.e. with $-1 \leq c_1(\mathcal{F}) \leq 0$) reflexive sheaf on T . If $d(c_1(\mathcal{F}) \cdot c_1(\mathcal{F})) > 4d(c_2(\mathcal{F}))$ then there exists $\alpha \leq 0$ such that $h^0(\mathcal{F}(\alpha)) \neq 0$. Furthermore, if $c_1(\mathcal{F}) = 0$ then $\alpha < 0$ hence we have $c_1(\mathcal{F}(\alpha)) < 0$.*

Proof. Let us denote by S the general (smooth) surface section of T . Since $\mathcal{F}|_S$ is a normalized rank 2 vector bundle on S , Bogomolov's theorem implies there exists $\alpha \leq 0$ such that $h^0(S, \mathcal{F}(\alpha)|_S) \neq 0$. Moreover, we can assume $\alpha < 0$ as soon

as $c_1(\mathcal{F}|_S) = 0$. Bogomolov's theorem implies that a section of $\mathcal{F}|_S(\alpha)$ can be chosen in such a way that it vanishes in the right dimension. In any case we have $c_1(\mathcal{F}|_S(\alpha)) < 0$, so Lemma 4.5 above implies $h^0(S, \mathcal{F}(\alpha)|_S) = 1$.

Fix $C \subset S$ a general curve section of T . We can assume that C does not meet the zero locus of the general section of $\mathcal{F}|_S(\alpha)$ so Lemma 4.5 implies:

$$(11) \quad h^0(C, \mathcal{F}(\alpha)|_C) = 1 \quad \text{and} \quad h^0(C, \mathcal{F}(\beta)|_C) = 0 \quad \forall \beta < \alpha.$$

Set

$$\mathcal{P} \simeq \mathbb{P}^{m-2} = \{\pi \in \mathbb{G}(n-m+2, \mathbb{P}^n) : C \subset \pi\} \subset \mathbb{G}(n-m+2, \mathbb{P}^n),$$

denote by $\tilde{T} \subset T \times \mathcal{P}$ the incidence variety:

$$\tilde{T} = \{(x, \pi) \in T \times \mathcal{P} : x \in \pi \cap T\}$$

and by $\phi : \tilde{T} \rightarrow T$, $\psi : \tilde{T} \rightarrow \mathcal{P}$ the natural maps.

Claim 1. $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) = 1, \quad \forall p \in \mathcal{P}$.

As we have just said $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) = 1$ for a very general $p \in \mathcal{P}$ so, by semicontinuity, $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) \geq 1, \quad \forall p \in \mathcal{P}$. In order to prove the Claim it is then sufficient to prove that $h^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) < 2, \quad \forall p \in \mathcal{P}$. Set $S' = \psi^{-1}(p)$ and assume by contradiction $h^0(S', \mathcal{F}(\alpha)|_{S'}) \geq 2$. From the short exact sequence

$$0 \rightarrow \mathcal{F}|_{S'}(\alpha-1) \rightarrow \mathcal{F}|_{S'}(\alpha) \rightarrow \mathcal{F}|_C(\alpha) \rightarrow 0$$

and taking into account (11) we get $h^0(S', \mathcal{F}(\alpha-1)|_{S'}) \neq 0$. Set $\bar{\alpha} := \min\{\beta \in \mathbb{N} : h^0(S', \mathcal{F}(\beta)|_{S'}) \neq 0\} \leq \alpha - 1$. From the short exact sequence

$$0 \rightarrow \mathcal{F}|_{S'}(\bar{\alpha}-1) \rightarrow \mathcal{F}|_{S'}(\bar{\alpha}) \rightarrow \mathcal{F}|_C(\bar{\alpha}) \rightarrow 0$$

and by the definition of $\bar{\alpha}$ we find $h^0(C, \mathcal{F}|_C(\bar{\alpha})) \neq 0$ which contradicts (11) since $\bar{\alpha} < \alpha$. The claim is so proved.

By Grauert's theorem [15, Corollary 12.9], $\psi_*\phi^*\mathcal{F}(\alpha)$ is an invertible sheaf on \mathcal{P} . On the other hand, since $\phi^{-1}C = C \times \mathcal{P}$, we have

$$\psi_*(\phi^*(\mathcal{F}(\alpha))|_{\phi^{-1}C}) \simeq H^0(C, \mathcal{F}(\alpha)|_C) \otimes \mathcal{O}_{\mathcal{P}} \simeq \mathcal{O}_{\mathcal{P}}.$$

Finally, the natural restriction $H^0(\psi^{-1}(p), \mathcal{F}(\alpha)|_{\psi^{-1}(p)}) \rightarrow H^0(C, \mathcal{F}(\alpha)|_C)$ is an isomorphism $\forall p \in \mathcal{P}$, so the natural map $\psi_*\phi^*(\mathcal{F}(\alpha)) \rightarrow \psi_*(\phi^*(\mathcal{F}(\alpha))|_{\phi^{-1}C}) \simeq \mathcal{O}_{\mathcal{P}}$ is an isomorphism of invertible sheaves on \mathcal{P} . Then we have

$$H^0(\tilde{T}, \phi^*(\mathcal{F}(\alpha))) = H^0(\mathcal{P}, \psi_*(\phi^*(\mathcal{F}(\alpha)))) = H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}}) = \mathbb{C}.$$

We conclude by means of the projection formula, because $\phi_*\mathcal{O}_{\tilde{T}} \simeq \mathcal{O}_T$. \square

Remark 4.7. (1) By Lemma 4.5, the coefficient α arising in Proposition 4.6 is the least twist of \mathcal{F} admitting a section.

(2) The proof of Proposition 4.6 shows that the zero locus of the section of $\mathcal{F}(\alpha)$ has the right dimension, because it does not meet the general curve C .

Proof of Theorem 4.1. In this proof we closely follow [18].

By Proposition 4.4 there exists a normalized reflexive sheaf \mathcal{F} (on T) such that

$$0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{F}(k) \rightarrow \mathcal{I}_X(e-t) \rightarrow 0$$

($c_1(\mathcal{F}) = cH$, $c_2(\mathcal{F}) = [X] - (ck + k^2)H^2$ and $c + 2k = e - t$). Set

- let α and β be the smallest degrees of two independent generators of $H_*^0 \mathcal{F}$ (compare with [18, p. 103]) ,
- $s = \min\{r : h^0(\mathcal{I}_{X,T}(r)) \neq 0\}$.

We distinguish two cases depending on whether the discriminant of \mathcal{F} is ≤ 0 or > 0 .

$d(c_1(\mathcal{F}) \cdot c_1(\mathcal{F})) \leq 4d(c_2(\mathcal{F}))$. This case is the simplest one because the expression $d(X) - d(T)h(e' - h - t)$ is the degree of the second Chern class of $\mathcal{F}(k - h)$. Since the discriminant is ≤ 0 , the second Chern class is always positive and we are done.

$d(c_1(\mathcal{F}) \cdot c_1(\mathcal{F})) > 4d(c_2(\mathcal{F}))$. In this case Proposition 4.6 implies $\alpha \leq 0$ (< 0 if $c = 0$). Furthermore, Remark 4.7 (2) says that the corresponding section vanishes in the right dimension. Then $d(c_2(\mathcal{F}(\alpha))) = d(c_2(\mathcal{F}(-\alpha - c))) \geq 0$ and the degree of the second Chern class is positive for any twist $\leq \alpha$ or $\geq -\alpha - c$. If $k = \alpha$ then $s = \beta + \alpha + c$ and the expression $d(X) - d(T)h(e' - h - t)$ is the degree of $c_2(\mathcal{F}(k - h)) = c_2(\mathcal{F}(h - \alpha - c))$ which is strictly positive since $h > 0$. So the inequality is proved and the equality cannot be attained. If $k \geq \beta$ then $s = \alpha + k + c$ and the expression $d(X) - d(T)h(e' - h - t)$ is the degree of $c_2(\mathcal{F}(k - h)) = c_2(\mathcal{F}(\alpha - (s - h)))$. So the inequality is proved and the equality can be attained only if $s = h$ and the degree of $c_2(\mathcal{F}(\alpha))$ vanishes. \square

Proof of Theorem 1.3 (1). It suffices to apply Theorem 4.1 to the complete intersection $T := F \cap G$, which is aCM with $\dim T = 3$ and $\omega_T \simeq \mathcal{O}_T(n + k - 6)$. The hypotheses that T is factorial and smooth in codimension 2 and that the very general surface section of T is factorial follow from Proposition 2.8. \square

REFERENCES

- [1] Beorchia, V. - Franco, D.: *On the moduli space of 't Hooft bundles*. Ann. Univ. Ferrara Sez. VII **47**, 253-268, (2001)
- [2] Di Gennaro, V. - Franco, D.: *A speciality for curves in \mathbb{P}^5* . Geom. Dedicata **129**, 89-99, (2007)
- [3] Di Gennaro, V. - Franco, D.: *Factoriality and Néron-Severi groups*. Commun. Contemp. Math. **10**, No 5, 745-764, (2008)
- [4] Di Gennaro, V. - Franco, D.: *Monodromy of a family of hypersurfaces*. Ann. Sci. Éc. Norm. Supér., 4^e série, **42**, No 3, , 517-529, (2009)
- [5] Di Gennaro, V. - Franco, D.: *Noether-Lefschetz Theory and Néron-Severi group*. Int. J. Math. **23**, No 1, Article ID 1250004, 12 p. (2012)
- [6] Di Gennaro, V. - Franco, D.: *Noether-Lefschetz theory with base locus*. Rend. Circ. Mat. Palermo (2) **63**, No 3, 257-276, (2014)
- [7] Ellia Ph. - Franco, D.: *codimension two subvarieties of \mathbb{P}^5 and \mathbb{P}^6* . J. Algebraic Geom. **11**, No 3, 513-533 (2002)
- [8] Ellia Ph. - Franco, D. - Gruson L.: *On subcanonical surfaces of \mathbb{P}^4* . Math. Z. **251**, No 2, 257-265 (2005)

- [9] Ellia Ph. - Franco, D. - Gruson, L.: *Smooth divisors of projective hypersurfaces*. Comment. Math. Helv. **83**, No 2, 371-385 (2008)
- [10] Flenner, H. - O'Carroll L. - Vogel W.: *Joins and intersections*. Springer-Verlag, 1999.
- [11] Franco, D. - Kleiman S. L. - Lascu, A.T.: *Gherardelli Linkage and Complete Intersections*. Mich. Math. J. **48**, Spec. Vol., 271-279, (2000)
- [12] Fulton, W.: *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete; 3.Folge, Bd. 2, Springer-Verlag 1984.
- [13] Gruson, L.-Peskin, Ch.: *Genre des courbes dans l'espace projectif*. Algebraic Geometry: Proceedings, Norway, 1977, Lecture Notes in Math., Springer-Verlag, New York 687, 31-59 (1978).
- [14] Gruson, L.-Peskin, Ch.: *Théorème de spécialité*. Astérisque, 71-72, 219-229 (1980)
- [15] Hartshorne, R., *Algebraic Geometry*. Graduate Texts in Mathematics, **52**, Springer Verlag (1977)
- [16] Hartshorne, R., *Stable reflexive sheaves*. Math. Ann. **254**, 121-176 (1980)
- [17] Peskin, Ch. - Spiro, L.: *Liaison des variétés algébriques*. Invent. Math. **26**, No 1, 271-302 (1974)
- [18] Roggero, M. - Valabrega, P.: *The speciality Lemma, rank 2 bundles and Gherardelli-type theorems for surfaces in \mathbb{P}^4* . Compositio Math. **139**, 101-111 (2003)
- [19] Serre, J. P.: *Algèbre Locale - Multiplicités* Springer LNM **11**, (1965)

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